Quantum Mechanics I

Week 3 (Solutions)

Spring Semester 2025

1 Compatible Observables

We consider the two observables ξ and η acting in a three-dimensional Hilbert space. In the standard basis $(|1\rangle, |2\rangle, |3\rangle)$ their matrix representation looks as follows:

$$\xi \to \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \qquad \eta \to \begin{pmatrix} 0 & \gamma & 0 \\ \gamma^* & 0 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, \tag{1.1}$$

where $\xi_1 \neq \xi_3$ and $\xi_2 \neq \xi_3$.

1. Under which conditions on the entries of the above matrix representations are the two observables compatible?

   By calculating the commutator we get:

   $$[\xi, \eta] = \xi\eta - \eta\xi = \begin{pmatrix} 0 & (\xi_1 - \xi_2)\gamma & 0 \\ (\xi_2 - \xi_1)\gamma^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2}$$

   Consequently, the two operators are only compatible (i.e., they commute) when $\xi_1 = \xi_2$ or $\gamma = 0$. In the following, we assume $\xi_1 = \xi_2$.

2. In the case in which the two observables are compatible: Find the basis that diagonalizes both observables simultaneously. Are the two observables a complete set of observables that commute?

   With $\xi_1 = \xi_2$, the two matrices commute: indeed, the first is diagonal and, when the second consists of a non-diagonal 2×2 block, the first is a multiple of the identity. To find the basis in which they are simultaneously diagonal, it suffices to diagonalize the 2×2 block: since its trace is 0 and the determinant is $-|\gamma|^2$, the eigenvalues are $\pm|\gamma|$ and, by putting $\gamma = |\gamma| e^{i\phi}$, the eigenstates are:

   $$|\tilde{1}\rangle \equiv \frac{1}{\sqrt{2}}(|1\rangle + e^{-i\phi}|2\rangle), \quad |\tilde{2}\rangle \equiv \frac{1}{\sqrt{2}}(|1\rangle - e^{-i\phi}|2\rangle).$$

   In the new basis $|\tilde{1}\rangle, |\tilde{2}\rangle, |\tilde{3}\rangle = |3\rangle$, the representation of the two observables is:

   $$\xi \to \begin{pmatrix} \xi & & \\ & \xi & \\ & & \xi_3 \end{pmatrix}, \quad \eta \to \begin{pmatrix} |\gamma| & & \\ & -|\gamma| & \\ & & \eta_3 \end{pmatrix}, \tag{1.3}$$

where $\xi \neq \xi_3$. The two observables form a complete set of commuting observables if there are no equal pairs of eigenvalues, i.e., if $(\xi_1, |\gamma|), (\xi_2, -|\gamma|), (\xi_3, \eta_3)$ are all different, i.e., if $\gamma \neq 0$.

Note: A pair of observables is called a complete set of commuting observables if these observables commute and we can identify a unique eigenvector (up to phase) for each set of eigenvalues of these operators.

3. Is it possible that an operator (a matrix) is at the same time unitary and Hermitian? If yes, give examples.

   From the conditions of unitarity $\hat{U}\hat{U}^\dagger = 1$ and hermiticity $\hat{U}^\dagger = \hat{U}$ of the operator, it follows that $\hat{U}^2 = 1$. An operator with the two eigenvalues ± 1 has such properties. Typical examples are the Pauli matrices, for which we have $\sigma_i^2 = 1$.

2 Quantum States for Spin-1/2 Particles

Note on the notation: For the spin operators, we will express any eigenvector as $|S_z; +\rangle$, where in the first entry we will have the relevant operator (to which this state corresponds to), and in the second entry the sign of the eigenvalue to which this ket corresponds to. Thus for the eigenvectors of $\hat{S}_z$, we have $|S_z; +\rangle$, $|S_z; -\rangle$.

A. The state representing the spin of an electron is given in the basis S_z as follows:

$$|\psi_1\rangle = |S_z; +\rangle + \sqrt{2}\,|S_z; -\rangle\,. \tag{2.1}$$

(i) Normalize the state $|\Psi_1\rangle$.

The normalization condition requires $\langle\Psi_1|\Psi_1\rangle = 1$. Thus, we have:

$$\langle\psi_1|\psi_1\rangle = 3$$

where we have used the orthonormality condition of the eigenvectors of S_z, i.e. $\langle S_z; i|S_z; j\rangle = \delta_{ij}$ where $i, j \in \{-1, +1\}$. Thus the new state is:

$$|\Psi_1\rangle = \frac{|\psi_1\rangle}{\sqrt{\langle\psi_1|\psi_1\rangle}} = \frac{1}{\sqrt{3}}\,|S_z; +\rangle + \sqrt{\frac{2}{3}}\,|S_z; -\rangle\,. \tag{2.2}$$

(ii) What is the probability to find the electron in a spin $|S_z; +\rangle$ state in a measurement of the S_z? Repeat for $|S_z; -\rangle$.

The probabilities are obtained from the coefficients

$$P_\pm = |\,\langle S_z; \pm|\Psi_1\rangle\,|^2\,. \tag{2.3}$$

Thus, we have $P_+ = 1/3$ and $P_- = 2/3$.

(iii) What is the expectation value of S_z?

The expectation value is computed by $\langle S_z \rangle = \langle \Psi_1 | S_z | \Psi_1 \rangle$ and found to be equal to $-\hbar/6$. We have used the orthonormality condition of the eigenvectors of S_z, i.e. $\langle S_z; i | S_z; j \rangle = \delta_{ij}$ where $i, j \in \{-1, +1\}$, and the action of S_z on its eigenvectors.

(iv) We would like to perform a measurement of S_x. To do this, express the state $|\Psi_1\rangle$ in the basis of S_x ($|S_x; \pm\rangle$), and calculate the probability of measuring $|S_x; +\rangle$ and $|S_x; -\rangle$.

The state $|\Psi_1\rangle$ expressed in the basis of S_x is:

$$|\Psi_1\rangle = |S_x; +\rangle \langle S_x; + | \Psi_1 \rangle + |S_x; -\rangle \langle S_x; - | \Psi_1 \rangle \,. \tag{2.4}$$

We need to compute the overlaps of the eigenvectors of S_x with S_z.

$$\langle S_x; + | S_z; + \rangle = \frac{1}{\sqrt{2}}, \quad \langle S_x; + | S_z; - \rangle = \frac{1}{\sqrt{2}},$$
$$\langle S_x; - | S_z; + \rangle = \frac{1}{\sqrt{2}}, \quad \langle S_x; - | S_z; - \rangle = -\frac{1}{\sqrt{2}} \,.$$

Then, using the above in Eq. (2.4), we find:

$$|\Psi\rangle = |S_x; +\rangle \frac{(1+\sqrt{2})}{\sqrt{6}} + |S_x; -\rangle \frac{(1-\sqrt{2})}{\sqrt{6}} \,. \tag{2.5}$$

The probabilities are:

$$P_+ = \frac{3+2\sqrt{2}}{6}, \quad P_- = \frac{3-2\sqrt{2}}{6} \,. \tag{2.6}$$

B. Given the following state expressed in the basis S_z:

$$|\Psi_2\rangle = \frac{1}{\sqrt{3}} |S_z; +\rangle + \frac{\sqrt{2} e^{i\pi/6}}{\sqrt{3}} |S_z; -\rangle \,. \tag{2.7}$$

(i) Verify that this state is normalized.

The normalization condition requires $\langle \Psi_2 | \Psi_2 \rangle = 1$. Thus, we have:

$$\langle \Psi_2 | \Psi_2 \rangle = \frac{1}{3} + \frac{2|e^{i\pi/6}|^2}{3} = 1 \,,$$

where we have used the orthonormality condition of the eigenvectors of S_z, i.e. $\langle S_z; i | S_z; j \rangle = \delta_{ij}$ where $i, j \in \{-1, +1\}$.

(ii) Give the matrix representation of the observable $S_\phi = \cos\phi S_x + \sin\phi S_y$.

Using $S_x = \hbar/2\sigma_x$ and $S_y = \hbar/2\sigma_y$, where σ_x, σ_y are Pauli matrices, we obtain:

$$S_\phi = \frac{\hbar}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \,. \tag{2.8}$$

(iii) Calculate the expectation value of S_ϕ for state $|\Psi_2\rangle$.

We compute the expectation value using the matrix form of the operator S_ϕ and the vector form of the state $|\Psi_2\rangle$,

$$\begin{aligned}\langle\Psi_2|S_\phi|\Psi_2\rangle &= \begin{pmatrix}\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}e^{-i\pi/6}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & e^{-i\phi}\\ e^{i\phi} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \sqrt{\frac{2}{3}}e^{i\pi/6}\end{pmatrix}\\ &= \frac{\hbar}{2}\begin{pmatrix}\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}e^{-i\pi/6}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2}{3}}e^{-i\phi}e^{i\pi/6}\\ \frac{1}{\sqrt{3}}e^{i\phi}\end{pmatrix}\\ &= \frac{\hbar}{2}\frac{\sqrt{2}}{3}\left(e^{i(\pi/6-\phi)}+e^{-i(\pi/6-\phi)}\right)\\ &= \frac{\hbar\sqrt{2}}{3}\cos(\pi/6-\phi).\end{aligned}$$

(iv) What is the uncertainty ΔS_ϕ given $|\Psi_2\rangle$? Note: $(\Delta S_\phi)^2 = \langle(S_\phi - \langle S_\phi\rangle)^2\rangle$.

For the uncertainty of S_ϕ, defined by $\langle(\Delta S_\phi)^2\rangle = \left\langle(S_\phi - \langle S_\phi\rangle)^2\right\rangle$, we use the identity $\left\langle(X-\langle X\rangle)^2\right\rangle = \langle X^2\rangle - \langle X\rangle^2$. Since $\left\langle S_\phi^2\right\rangle = \frac{\hbar^2}{4}$, we simply have

$$\left\langle(\Delta S_\phi)^2\right\rangle = \frac{\hbar^2}{4} - \frac{\hbar^2}{9}2\cos^2(\pi/6-\phi) = \frac{\hbar^2}{4}\left(1-\frac{8}{9}\cos^2(\pi/6-\phi)\right).$$

(v) Calculate the uncertainty of the observable $S_\theta = \cos\theta S_z + \sin\theta S_x$ given the state $|S_z;+\rangle$. Does the result meet your expectations?

We can write the average value in terms of the dot product :

$$\langle S_z;+|S_\theta|S_z;+\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\cos\theta. \tag{2.9}$$

The uncertainty is calculated as usual:

$$\left\langle(\Delta S_\theta)^2\right\rangle = \left\langle S_\theta^2\right\rangle - \langle S_\theta\rangle^2 = \frac{\hbar^2}{4}\sin^2\theta\,. \tag{2.10}$$

The expectation value is taken in the state $|S_z;+\rangle$.

The S_θ operator performs a θ angle rotation. Therefore, the overlap between the initial state $|S_z;+\rangle$ and the rotated state depends on the angle of rotation. As expected, we observe that for an angle rotation $\theta = \frac{\pi}{2}$, the states $S_\theta|S_z;+\rangle = |S_z;-\rangle$ and $|S_z;+\rangle$ are orthogonal and the mean value is 0. The uncertainty is maximized in this case. When $\theta = 0$, then the mean value is $\hbar/2$ as expected since $S_\theta|S_z;+\rangle = |S_z;+\rangle$, and the uncertainty is zero.

3 Change of Spin Basis

Consider the operators $\hat{A}, \hat{B}$ with

$$\hat{A} |a_i\rangle = a_i |a_i\rangle , \quad \hat{B} |b_i\rangle = b_i |b_i\rangle \ . \tag{3.1}$$

We are interested in finding the unitary matrix U that allows for a change of basis,

$$|b_i\rangle = U_{ab} |a_i\rangle \ . \tag{3.2}$$

where $|a_i\rangle , |b_i\rangle$ are the eigenvectors in the old and new basis, respectively.

(a) Consider $\hat{A} = \hat{S}_z$ and $\hat{B} = \hat{S}_x$. Find the unitary matrix $\hat{U}$.

The eigenvectors of $S_x = \frac{\hbar}{2}\sigma_x$ are (in the basis where S_z is diagonal)

$$|S_x; +\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad |S_x; -\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

We can find the transformation matrix which connects the eigenvectors of S_z and S_x.

$$U_{xz} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

The elements of this matrix are obtained in the S_z representation by multiplying Eq. (3.2) (for the case of the spin states) with $\langle S_z; j|$:

$$\langle S_z; j| U_{xz} |S_z; i\rangle = \langle S_z; j|S_x; i\rangle ,$$

where $i, j \in \{\pm\}$.

(b) This transformation may now be written as:

$$U = \sum_j |b_j\rangle \langle a_j| \ . \tag{3.3}$$

Confirm that you obtain the correct result when you apply this unitary transformation to the old basis eigen-kets.

We can immediately find that the transformation matrix can be written as

$$U_{xz} = |S_x; +\rangle \langle S_z; +| + |S_x; -\rangle \langle S_z; -| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} , \tag{3.4}$$

and we can verify that this is the same transformation matrix we got in the previous question.

(c) Repeat Questions (a,b) for $\hat{A} = \hat{S}_z$ and $\hat{B} = \hat{S}_y$.

Given that the eigenvalues of S_y are

$$|S_y;+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, \quad |S_y;-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix},$$

it can be found that

$$U_{yz} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$$

where

$$\langle S_z;j|\, U_{yz}\, |S_z;i\rangle = \langle S_z;j|S_y;i\rangle \ ,$$

where $i, j \in \{\pm\}$. The transformation matrix can be written as

$$U_{yz} = |S_y;+\rangle\langle S_z;+| + |S_y;-\rangle\langle S_z;-| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}.$$

(d) Find the representation of S_y in the eigenbasis of S_x.

We are looking for a unitary transformation U which diagonalizes S_x ($US_xU^\dagger = S_z$). We can take for example $U = U_{xz}$ and so we have

$$S_y \to \bar{S}_y = U_{xz}S_yU_{xz}^\dagger = \frac{\hbar}{2}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} = -S_y, \quad S_x \to \bar{S}_x = S_z, \quad S_z \to \bar{S}_z = S_x.$$

One can verify that the matrices $\bar{S}_x, \bar{S}_y, \bar{S}_z$ satisfy the usual spin commutation relation.

4 Sequential Stern-Gerlach Experiment

A beam of spin-1/2 atoms goes through a sequence of Stern-Gerlach experiments as follows:

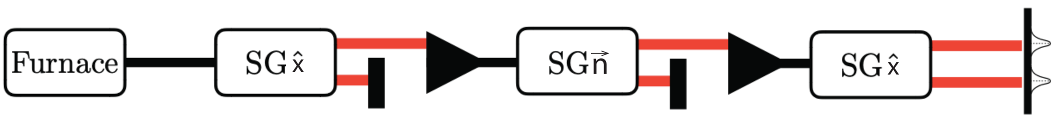

Figure 1: A sequential Stern-Gerlach Experiment

The measurement scheme is as follows:

i) The first measurement accepts atoms with $S_x = \hbar/2$ and rejects atoms with $S_x = -\hbar/2$.

ii) The second measurement accepts atoms with spin $S_{\vec{n}} = \hbar/2$ and rejects atoms with $S_{\vec{n}} = -\hbar/2$, where $\vec{n}$ is some unit vector and $S_{\vec{n}}$ are the eigenvalues of the operator $\hat{S}_{\vec{n}} = \vec{n} \cdot \vec{S}$.

iii) The third and final measurement is along the x-axis, and the distribution of each eigenvectors is projected on the screen.

We aim to determine the outcome on the screen. We will proceed with this in steps. First, we need to establish the orthonormal basis for each observable to be measured.

(a) Diagonalize the $\hat{S}_x$ matrix. Find the eigenvalues and eigenvectors $|S_x; \pm\rangle$ of this matrix.

The matrix representation of $\hat{S}_x$ is known

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

We apply the diagonalization procedure and find the eigenvalues as $\pm\hbar/2$ with corresponding eigenvectors

$$|S_x; +\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |S_x; -\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.2}$$

(b) Now, consider the following operator which expresses the projection of the spin operator $\vec{S}$ along a unit vector $\vec{n}$

$$\hat{S}_{\vec{n}} = \vec{n} \cdot \vec{S}, \tag{4.3}$$

where the unit vector $\hat{n}$ is expressed in spherical coordinates as $\vec{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$ and the spin operator is $\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$. The angles ϕ and θ correspond to the azimuthal and polar angles, respectively, as defined in the spherical coordinate system. Find the matrix form of this operator.

We use the matrix representation of each component of the spin operator:

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x, \quad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y, \quad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z, \tag{4.4}$$

where $\hat{\sigma}_j$, $j \in \{x, y, z\}$ are the Pauli matrices.

Then the matrix form of the operator $\hat{S}_{\vec{n}}$ reads

$$\hat{S}_{\vec{n}} = \frac{\hbar}{2}(\hat{\sigma}_x \sin\theta\cos\phi + \hat{\sigma}_y \sin\theta\sin\phi + \hat{\sigma}_z\cos\theta) = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}. \tag{4.5}$$

(c) Find the eigenvalues and eigenvectors of this matrix. Show that these are:

$$|S_{\vec{n}}; +\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \quad |S_{\vec{n}}; -\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}e^{i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{4.6}$$

Diagonalizing the matrix $\hat{S}_{\vec{n}}$ we find the following eigenvalues $\pm\hbar/2$. The corresponding normalized eigenvectors $|S_{\vec{n}}; \pm\rangle$ are found in the usual way, under the constraint of normalization.

(d) For the rest of this exercise, we consider a unit vector $\hat{n}$ along the xy-plane. What form does the operator $\vec{S}_{\vec{n}}$ take, and consequently its eigenvectors?

A unit vector on the xy-plane corresponds to $\theta = \pi/2$ and varying azimuthal angle ϕ, thus $\vec{n} = \cos\phi\hat{x} + \sin\phi\hat{y}$. The matrix form of $\hat{S}_{\vec{n}}$ is reduced to:

$$\hat{S}_{\vec{n}} = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} . \tag{4.7}$$

The corresponding eigenvectors simply become:

$$|S_{\vec{n}};+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}, \quad |S_{\vec{n}};-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{i\phi} \\ 1 \end{pmatrix} . \tag{4.8}$$

Having established the orthonormal bases that we will need to describe our experiment, let us consider the initial state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|S_x;+\rangle + |S_x;-\rangle \Big) . \tag{4.9}$$

(e) What are the intensities of the final beams for the outcomes $S_x = \pm\hbar/2$ if the outgoing beam from the first measurement is normalized to 1? You answer should be expressed in terms of the relevant spherical angle from Question (d).

The outgoing beam from the first measurement is $|\psi\rangle = |S_x;+\rangle$, normalized to unity. Then, after the second measurement, we must project $|\psi\rangle$ onto the eigenbasis of $S_{\vec{n}}$, and we do this by using the resolution of identity:

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \sum_j |S_{\vec{n}};j\rangle\langle S_{\vec{n}};j|\psi\rangle = |S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|\psi\rangle + |S_{\vec{n}};-\rangle\langle S_{\vec{n}};-|\psi\rangle . \tag{4.10}$$

After the second measurement, we retain $|\psi'\rangle = |S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|\psi\rangle$. Then, after the third measurement, we once again project onto the S_x basis, i.e.

$$\begin{aligned} |\psi'\rangle &= \mathbb{1}|\psi'\rangle = \\ &= \sum_j |S_x;j\rangle\langle S_x;j|\psi'\rangle = \\ &= |S_x;+\rangle\langle S_x;+|\psi'\rangle + |S_x;-\rangle\langle S_x;-|\psi'\rangle = \\ &= |S_x;+\rangle\langle S_x;+|S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|\psi\rangle + |S_x;-\rangle\langle S_x;-|S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|\psi\rangle . \end{aligned}$$

Replacing $|\psi\rangle = |S_x;+\rangle$, we get:

$$|\psi'\rangle = |S_x;+\rangle\langle S_x;+|S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|S_x;+\rangle + |S_x;-\rangle\langle S_x;-|S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|S_x;+\rangle .$$

The intensities of the outgoing beams are taken as the coefficients (squared-modulus) of the final state:

$$I_+ = |\langle S_x;+|S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|S_x;+\rangle|^2, \quad I_- = |\langle S_x;-|S_{\vec{n}};+\rangle\langle S_{\vec{n}};+|S_x;+\rangle|^2 . \tag{4.11}$$

To find the intensities, we now compute the following overlaps

$$\langle S_x;+|S_{\vec{n}};+\rangle = \frac{1}{\sqrt{2}}\Big[\langle S_z;+| + \langle S_z;-|\Big]\cdot\frac{1}{\sqrt{2}}\Big[|S_z;+\rangle + e^{i\phi}\,|S_z;-\rangle\Big] = \frac{1}{2}(1+e^{i\phi})$$
$$\langle S_x;-|S_{\vec{n}};+\rangle = \frac{1}{\sqrt{2}}\Big[\langle S_z;+| - \langle S_z;-|\Big]\cdot\frac{1}{\sqrt{2}}\Big[|S_z;+\rangle + e^{i\phi}\,|S_z;-\rangle\Big] = \frac{1}{2}(1-e^{i\phi})$$

Using the overlaps in the expressions for the intensities, we find

$$I_+ = \cos^4\frac{\phi}{2}, \quad I_- = \sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}\,.$$

In the above we used trigonometric identities $\cos\phi = 2\cos^2\frac{\phi}{2} - 1$ and $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$.

(f) What is the best choice for the spherical angle to maximize the intensity of the beam with $S_x = +\hbar/2$?

The best choice will be to choose $\phi = 0$ (in the range $\phi \in [0, 2\pi[$), for which we have $I_+ = 1, I_- = 0$. Notice that this case corresponds to $S_{\vec{n}} = S_x$, and thus both the second and third measurements are taken along S_x. Consequently, the probability of measuring $|S_x;+\rangle$ at the end will be one.

(g) What is the best choice for the spherical angle to maximize the intensity of the beam with $S_x = -\hbar/2$?

The best choice will be to choose $\phi = \pi/2, 3\pi/2$. These choices correspond to $I_- = I_+ = 1/4$.